Discrete Mathematics 312 (2012) 2009–2022

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Intrinsically linked signed graphs in projective space[☆]

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ARTICLE INFO

Article history:
Received 30 October 2010
Received in revised form 16 March 2012
Accepted 19 March 2012
Available online 9 April 2012

Keywords:
Projective space
Intrinsically linked
Signed graph

ABSTRACT

We define a signed embedding of a signed graph into real projective space to be an embedding such that an embedded cycle is 0-homologous if and only if it is balanced. We characterize signed graphs that have a linkless signed embedding. In particular, we exhibit 46 graphs that form the complete minor-minimal set of signed graphs that contain a non-split link for every signed embedding. With one trivial exception, these graphs are derived from different signings of the seven Petersen family graphs.

1. Introduction

Recall that a graph is *intrinsically linked* if every embedding of the graph in $\mathbb{R}^3$ contains at least two non-splittably linked cycles. The set of all minor-minimal intrinsically linked graphs is given by the seven Petersen family graphs [3,11,12]. These graphs are obtained from K_6 by $\Delta - Y$ and $Y - \Delta$ exchanges. We denote these by K_6, P_7, P_8, P_9, P_{10}, $K_{4,4} \setminus e$, and $K_{3,3,1}$, where P_{10} is the classic Petersen graph (see Appendix A).

We say that a graph is *intrinsically linked in* $\mathbb{R}P^3$ provided every embedding into $\mathbb{R}P^3$ contains at least one pair of non-splittably linked cycles (see the next section for a formal definition of non-splittably linked cycles in $\mathbb{R}P^3$). Note that every spatially embedded graph corresponds to a projectively embedded graph. We also note that some intrinsically linked spatial graphs have linkless embeddings in $\mathbb{R}P^3$ (for example, K_6, see [2]). As a result of Robertson and Seymour's Minor Theorem [10], the collection of minor-minimal intrinsically linked graphs in $\mathbb{R}P^3$ is finite. Bustamante et al., [2] fully characterized intrinsically linked graphs in $\mathbb{R}P^3$ with connectivity 0, 1, and 2 and in all, 597 graphs were shown to be minor-minimal intrinsically linked in $\mathbb{R}P^3$, although a complete classification is still unknown.

A *signed embedding* of a signed graph $\Sigma = (\Gamma, \sigma)$ is an embedding into $\mathbb{R}P^3$ for which a cycle is 1-homologous if and only if its sign is negative. An *intrinsically linked signed graph* denotes a signed graph for which every signed embedding contains at least one pair of non-splittably linked cycles. In this paper, we seek to classify all minor-minimal intrinsically linked signed graphs in $\mathbb{R}P^3$. This is motivated by Zaslavsky's approach for projective planar graphs [15]. Zaslavsky found

☆ This material is based upon work supported by the National Science Foundation under Grant No. DMA-0353050, and the National Security Administration under Grant No. H98230-09-1-0097.

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doi:10.1016/j.disc.2012.03.025

eight signed forbidden minors for projective planarity, significantly less than the set of 35 unsigned forbidden minors for projective planarity found by Archdeacon [1] and Glover et al. [5]. We seek the analogous set of graphs for intrinsically linked graphs in projective space, $\mathbb{R}P^3$, and have found a complete set of 46 such graphs. These 46 graphs are V_2^o, the seven balanced Petersen family graphs, 32 Petersen family graphs with a balancing vertex, and six Petersen family graphs that have all positive edges except for one 3-cycle with all edges negative.

1.1. Definitions and notation

Throughout, let M be a 3-manifold. We define a *graph* G as a set of vertices $V(G)$ and edges $E(G)$, where an *edge* is an unordered pair (v_i, v_j) with $v_i, v_j \in V$. It is not necessary that $i \neq j$. Our graphs are finite and are allowed to have loops and redundant edges. Let $v_1, v_2, \ldots, v_n \in V(G)$, with $n \geq 3$, and let

$$(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1) \in E(G)$$

such that $v_i \neq v_j$ for $i \neq j$, then the sequence of vertices $v_1, v_2, \ldots, v_n$ and edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)$ is an *n-cycle* in the graph G. A *loop* at vertex v is an edge of the form (v, v).

A *link* is two or more disjointly embedded circles (cycles) in M. Let $L_i \cup L_j$ be a 2-component link, where $i \neq j$. We say $L_i \cup L_j$ is *splittable* if there exists $A \subseteq M$, where A is homeomorphic to B^3, such that $L_i \subseteq A$ and $L_j \subseteq A^C$. Otherwise, $L_i \cup L_j$ is a *non-splittable* link. If every embedding of G into M contains a pair of cycles that form a non-splittable two-component link, then G is *intrinsically linked* in M. In this paper, we will use "intrinsically linked" to mean intrinsically linked in $\mathbb{R}P^3$. Similarly, when we say "embedded" (respectively, "embedding") we mean "embedded (respectively, embedding) in $\mathbb{R}P^3$".

If H is a graph such that H can be obtained from G by a sequence of edge removals, vertex removals, and edge contractions, then H is a *minor* of G, written $H \leq G$. If $H \leq G$ but $H \neq G$, then H is a *proper minor* of G, written $H < G$. If $e \in E(G)$ is the edge contracted or deleted in G to obtain H, we write that $H = G/e$ or $H = G \setminus e$, respectively. To contract an edge $e = (v, w)$, replace e with the new vertex v_e, which becomes adjacent to all of the former neighbors of v and w. Note that if H, a minor of a graph G, is intrinsically linked, then G is as well [9].

A *signed graph* $\Sigma = (G, \sigma)$ consists of a graph G and an edge signing $\sigma : E(G) \to \{+, -\}$. We also denote G as $|\Sigma|$. The sign of a cycle $C = (v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1) = e_1e_2 \cdots e_{n-1}e_n$ is obtained by multiplying the signs of its constituent edges:

$$\sigma(C) = \sigma(e_1)\sigma(e_2)\cdots\sigma(e_n).$$

The sign of a path is defined similarly. A signed graph Σ is *balanced* if all of its cycles are positive. Otherwise, it is *unbalanced*.

Recall that *switching* a signed graph Σ means reversing the signs of all edges between a vertex set $X \subseteq V(\Sigma)$ and its complement. We say Σ_1 and Σ_2 are *switching isomorphic* if one can be obtained from the other by a sequence of switchings. Recall that switching preserves signs of cycles, and two signed graphs are switching isomorphic if and only if all signs of corresponding cycles are the same. We sometimes abuse notation by using Σ in the remainder of the paper to denote a particular signed graph, as well as the switching class of Σ. It is not difficult to see that every balanced graph Σ is switching isomorphic to an all positive signing of its underlying graph, which we denote Σ^+. (Similarly, an all negative signing of Σ is denoted Σ^-.) A vertex v of a signed graph Σ is a *balancing vertex* of Σ if $\Sigma \setminus v$ is balanced, but Σ is not.

Contracting an edge on a signed graph is only allowed after the graph is switched so that the edge being contracted is positive. This is necessary in order to preserve the sign of each cycle. A *minor of a signed graph* $\Sigma = (G, \sigma)$ is any signed graph that can be obtained from Σ by a sequence of switchings, vertex deletions, edge deletions, and edge contractions. Note that Zaslavsky [15,14] used the term "link minor" to denote our version of "minor". We use the shorter term so as not to overuse the term "link".

Recall that *real projective space*, denoted $\mathbb{R}P^3$, is the sphere S^3 with identified opposite points, or equivalently as the closed 3-ball B^3 with antipodal boundary points identified. All of our embedded graphs and ambient isotopies are in the PL category, which is possible as $\mathbb{R}P^3$ is a quotient of S^3. An embedded cycle is 0-*homologous* (0-H) if it crosses the boundary an even number of times and 1-*homologous* (1-H) if it crosses the boundary an odd number of times. We say that an embedded graph is *affine* if it is contained in a 3-ball in $\mathbb{R}P^3$.

An important signed graph, V_2^o, is the graph consisting of two disjoint loops, each signed negatively.

2. Preliminary results

Lemma 1. *If Σ has a linkless embedding, then so does every minor of Σ.*

Proof. Embed Σ linklessly in projective space. Consider an edge e. If necessary, switch so that e has positive sign (this corresponds, geometrically, to performing ambient isotopy so that in the embedding, e does not touch the boundary). Contract the edge e to obtain a linkless embedding of Σ/e. Similarly, removing an edge or a vertex results in a linkless embedding. □

It follows [10] that we can indeed characterize the set of minor-minimal intrinsically linked signed graphs by a finite set of minor-minimal graphs with this property. We exhibit one such graph now.

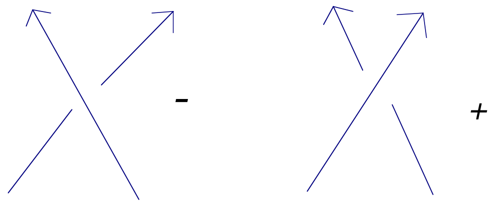


Fig. 1. Signs of oriented crossings.

Proposition 2. *The graph V_2^o is minor-minimal intrinsically linked.*

Proof. Two disjoint 1-homologous cycles embedded in $\mathbb{R}P^3$ cannot form a non-splittable link, as being able to place one component in a 3-ball would imply that the component is 0-homologous. Clearly V_2^o is minor-minimal. □

2.1. Balanced minor-minimal graphs

As could be expected, the minor-minimal intrinsically linked signed graphs are closely related to the set of minor-minimal intrinsically linked graphs in $\mathbb{R}^3$, that is, the Petersen family graphs.

The following technical lemma will help us analyze balanced Petersen family graphs. Given two (vertex) disjoint edges E_1 and E_2 in a graph G, we say that E_1 and E_2 *extend* to the (vertex) disjoint cycle pair (C_1, C_2) if $E_i \subset C_i$ for $i = 1, 2$.

Lemma 3. *Let H be obtained from G by a $\Delta - Y$ exchange. If every pair of disjoint edges in G extends to an even number of disjoint cycle pairs, then every pair of disjoint edges in H also extends to an even number of disjoint cycle pairs.*

Proof. Assume that every pair of disjoint edges in G extends to an even number of disjoint cycle pairs. Now insert the vertex v and connect it to the triangle vertices, v_1, v_2, and v_3, to obtain H from G. Any given pair of disjoint edges of H edge disjoint from the Y extend to the same number of disjoint cycle pairs as they did in G, replacing any edge of the form (v_i, v_j) in such a cycle by a path (v_i, v, v_j).

Consider two disjoint edges in H such that one is from the Y and the other shares exactly one vertex with the Y. Without loss of generality, we may take our edges to be (v_1, v) and (v_2, w). In G, the disjoint edges (v_1, v_3) and (v_2, w) extend to an even number of cycle pairs. Every cycle that includes the edge (v_1, v_3) in G corresponds to one that includes the path (v_1, v, v_3) in H. A cycle that includes (v_2, w) in G, disjoint from (v_1, v_3), corresponds to a cycle with the same vertices in H, that is disjoint from (v_1, v, v_3). Hence in H, (v_1, v) and (v_2, w) extend to the same number of disjoint cycle pairs as (v_1, v_3) and (v_2, w) did in G, which is an even number.

Finally, consider two disjoint edges in H where one is an edge of the Y and the other, e, is vertex disjoint from the Y. Without loss of generality, take (v_1, v) to be the edge in the Y. Recall that in G, the disjoint edges (v_1, v_2) and e extend to $2k$ cycle pairs, and (v_1, v_3) and e also extend to an even number of cycle pairs, say $2j$. In H, (v_1, v) is in every cycle that corresponds to a cycle in G that contains either (v_1, v_2) or (v_1, v_3), except for the cycle (v_1, v_2, v_3). Note that the cycle (v_1, v_2, v_3) is counted the same number of times, say n, in both the (v_1, v_2) and (v_1, v_3) cycle pairs. Then (v_1, v) and e extend to $2k + 2j - 2n$ cycle pairs, which is even. □

Recall that the classic notion of linking number extends to links in $\mathbb{R}P^3$. Suppose L and K are circles embedded in $\mathbb{R}P^3$; orient L and K. In a projection of L and K, at each crossing, assign a $+1$ or -1 as shown in Fig. 1. Then the *linking number of L and K*, $lk(L, K)$ is the sum of the numbers, $+1$ or -1 at each crossing in the projection, divided by 2. In $\mathbb{R}P^3$ there are five generalized Reidemeister moves [4], see also [6]. One can use Reidemeister moves to justify that linking number is well-defined. In particular, the linking number of a splittable 2-component link is 0. Given a signed embedding of $\Sigma = (G, \sigma)$,where G is a Petersen family graph, we define $LK(\Sigma) = LK(G, \sigma)$ to be $\sum lk(C_i, C_j)$ mod 2, summed over all pairs of disjoint cycles in G.

The hypothesis of the preceding lemma applies to K_6 [3,12] and thus conclusion of the lemma applies to every graph in the Petersen family. It follows that an arbitrary crossing change between disjoint edges in a projection of a Petersen family graph embedded in $\mathbb{R}P^3$ changes by 1 or -1 the linking number of an even number of cycle pairs, and hence that such a crossing change will not change $LK(G, \sigma)$. Thus the definition of $LK(G, \sigma)$ is independent of embedding. This leads us to the next proposition.

Proposition 4. *The balanced minor-minimal intrinsically linked graphs are K_6^+, $K_{3,3,1}^+$, P_7^+, P_8^+, P_9^+, P_{10}^+, and $K_{4,4} \setminus e^+$.*

Proof. Let $\Sigma = (\Gamma, \sigma)$ be balanced, and suppose it does not contain a balanced Petersen Family graph as a minor. Then by [11], Γ can be embedded linklessly in $\mathbb{R}^3$, and so Σ has an affine linkless embedding in $\mathbb{R}P^3$. Now assume Γ does contain a Petersen Family graph as a minor, G, and embed Σ in $\mathbb{R}P^3$. Contract and delete edges and vertices to obtain an embedding of G in $\mathbb{R}P^3$. Then we can obtain an affine embedding of G using ambient isotopy and crossing changes, (which can be done, thanks to the diagrammatic unknotting result of Mroczkowski [8]). By Lemma 3, the crossing changes preserve non-zero $LK(G, \sigma)$. Thus the original embedding of Σ must have contained a non-split link, and so Σ is intrinsically linked. □

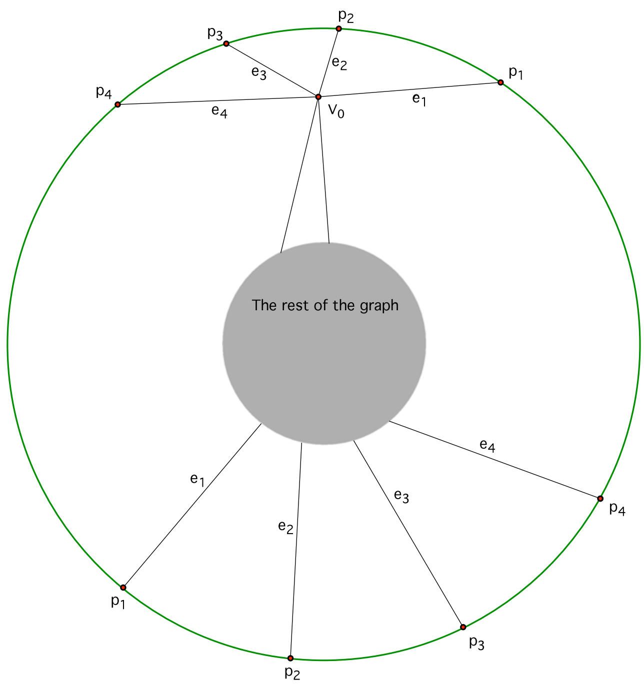


Fig. 2. An embedding of Σ, where v_0 is a balancing vertex and $\{e_1, e_2, e_3, e_4\}$ are the negative edges.

Corollary 5. *Let a signed graph Σ be balanced. If Σ has a linkless embedding, then Σ has an affine linkless embedding.*

Proof. Let Σ be a balanced graph with a linkless embedding. Then by Proposition 4, Σ contains no Petersen family graph as a minor. As the Petersen family graphs form the complete set of minor-minimal intrinsically linked graphs in $\mathbb{R}^3$, Σ must have an affine linkless embedding. □

2.2. Graphs with a balancing vertex

Lemma 6. *For a signed Petersen family graph Σ with a balancing vertex, $LK(\Sigma) = LK(\Sigma^+)$.*

Proof. Let Σ be a signed graph with a balancing vertex. We may switch so that all n negative edges $e_1, \dots, e_n$ contain the balancing vertex, v_o, as an endpoint. Consider an embedding of Σ. Using ambient isotopy and crossing changes, we may deform the embedding so that v_o is near, but not on, the boundary; all negative edges cross the boundary once and all positive edges are disjoint from the boundary. Further, we may assume that no negative edges cross one another in a projection (see Fig. 2).

Let $p_1, \dots, p_n$ be points on the edges $e_1, \dots, e_n$ that intersect the boundary of the projection (see Fig. 2). Let $U_1, \dots, U_n$ be sufficiently small neighborhoods of $p_1, \dots, p_n$ in the projection such that, for each U_i, its corresponding edge e_i intersects ∂U_i in only two points, p_i' and q_i', and no other edge e_j intersects ∂U_i assuming $j \neq i$. Let $q_1', \dots, q_n'$ be the points closest to v_0 and $p_1', \dots, p_n'$ correspond antipodally.

Connect each $p_1', \dots, p_n'$ with v_0 with edges $e_1', \dots, e_n'$, not crossing the boundary, to obtain an embedding of the graph Σ^+. Finally, delete the regions of $e_1, \dots, e_n$ within each ∂U_i, respectively (see Fig. 3), and smooth out the new degree 2 vertices $p_1', \dots, p_n'$. Note that every $1 - H$ cycle γ in Σ corresponds to exactly one $0 - H$ cycle γ' in Σ^+. Also note that for every cycle disjoint from γ in Σ, δ, there is a corresponding cycle, δ' in Σ^+ with $LK(\gamma, \delta) = LK(\gamma', \delta')$.

Note also that it is possible to embed the edges $e_1', \dots, e_n'$ without crossing any pre-existing edges in Σ^+ other than $e_1, \dots, e_n$, which no longer exist in Σ^+. Further, e_i' will not cross e_j'. Since no new crossings are added and ambient isotopy and crossing changes preserve $LK(P, \sigma)$ for a graph in the Petersen family (by Lemma 3), it follows that $LK(\Sigma) = LK(\Sigma^+) = 1$. □

The procedure outlined in the previous proof, essentially done in reverse, establishes the following proposition:

Proposition 7. *If G has a linkless spatial embedding and σ is a signing of G for which there exists a balancing vertex, then (G, σ) has a linkless embedding for which the only edges that cross the boundary are incident to the balancing vertex.*

Proof. Start with an affine linkless embedding of G. Use ambient isotopy to make a projection so that edges incident to the (to be) balancing vertex v_0 are positioned as in Fig. 3. Then alter the edges so that the embedding in Fig. 2 results, to get

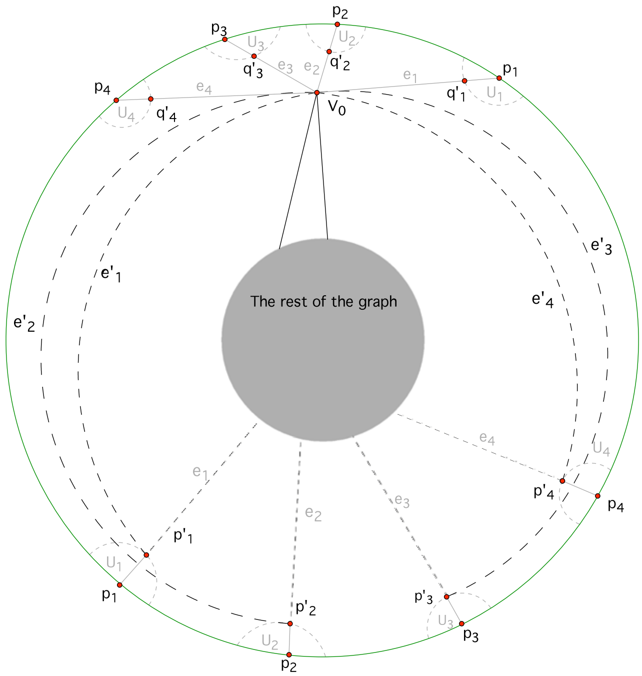


Fig. 3. The corresponding affine embedding of Σ^+.

the desired embedding of (G, σ). Note that each 0-homologous cycle in the resulting embedding is splittable from all other disjoint cycles. It follows that the resulting embedding is linkless. □

From the previous lemma and corollary, we can add to our list of signed minor-minimal intrinsically linked signed graphs.

Proposition 8. *A graph Σ with a balancing vertex is minor-minimal intrinsically linked in $\mathbb{R}P^3$ if and only if $|\Sigma|$ is a Petersen family graph.*

Choosing different vertices as the balancing vertex yields 32 different signed graphs with a Petersen family graph as the underlying graph. We illustrate these in Appendix B.

3. Other sign classes of Petersen family graphs

There are two more possible signings of Petersen family graphs, one which results in an intrinsically linked graph and one which results in a graph with a linkless embedding. We discuss the intrinsically linked case first. We say that a signed graph *contains exactly one all-negative* 3*-cycle* provided all edges are positive except all edges of a 3-cycle are negative. For brevity, in figures we also denote this case K_3^-.

Lemma 9. *A signed Petersen family graph Σ containing exactly one all-negative* 3*-cycle is minor-minimal intrinsically linked.*

Proof. Let Σ be a signed Petersen family graph containing exactly one all-negative 3-cycle with edges e_1, e_2, and e_3. Starting from an arbitrary embedding of this graph into $\mathbb{R}P^3$, one can use ambient isotopy and crossing changes so that e_1, e_2, and e_3 only cross the boundary once and the rest of the edges will not cross the boundary, with $LK(\Sigma)$ preserved.

Let p_1, p_2, p_3 be the points at which edges e_1, e_2, e_3 cross the boundary, respectively. Identify neighborhoods U_1, U_2, U_3 around points p_1, p_2, p_3 such that no other edge e_j intersects U_i (where $i \neq j$) and e_i intersects U_i at only two points, p'_i and q'_i (see Fig. 4).

Connect each p'_1, p'_2, p'_3 with each q'_1, q'_2, q'_3 with edges e'_1, e'_2, e'_3 so that they do not cross each other, and leave all other edges unchanged. Finally, delete the regions of e_1, e_2, e_3 within each ∂U_i, respectively (see Fig. 5). The result is an embedding of the signed graph Σ^+. Since no crossings were introduced or removed, it follows that $LK(\Sigma) = LK(\Sigma^+) = 1$. Therefore, Σ is intrinsically linked. We further claim that an arbitrary minor of Σ, Σ_0, has a linkless embedding. Either Σ_0 has a balancing vertex, or it has an all-negative 3-cycle. If Σ_0 has a balancing vertex, then it has a linkless embedding by Proposition 8. Otherwise, we use the fact that $|\Sigma_0|$, a minor of a Petersen family graph, has a flat spatial embedding [11] (that is, every cycle C' bounds a disk D with $\Sigma_0 \cap D = \partial D = C'$). Thus, Σ_0^+ has a signed embedding with a projection as pictured in Fig. 5, where (A, B, C) forms the all-negative 3-cycle in Σ_0. By reversing the process described above, a linkless embedding of Σ_0 results. □

We remark further that in the case that Σ contains exactly one all-negative 3-cycle, Σ^+ cannot be switched to obtain Σ, since Σ has a negative cycle, nor can Σ be switched to have a balancing vertex, as Σ has no balancing vertex.

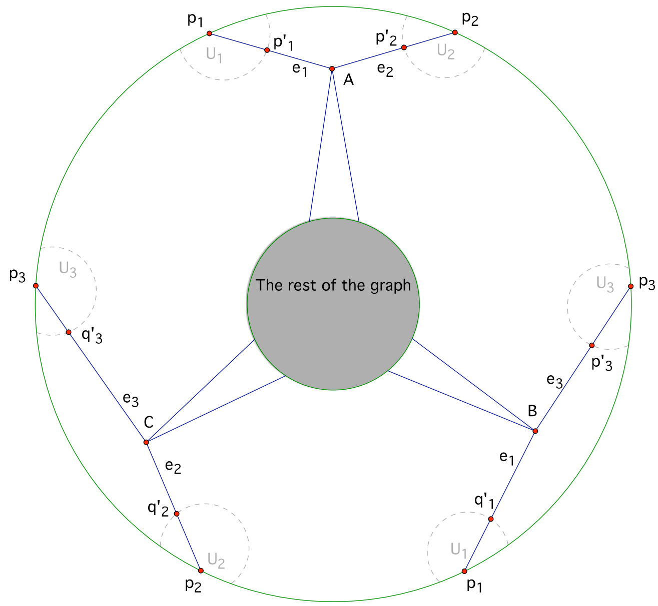


Fig. 4. A non-affine embedded Petersen family graph with 3 negative triangle edges and the neighborhoods near the boundary.

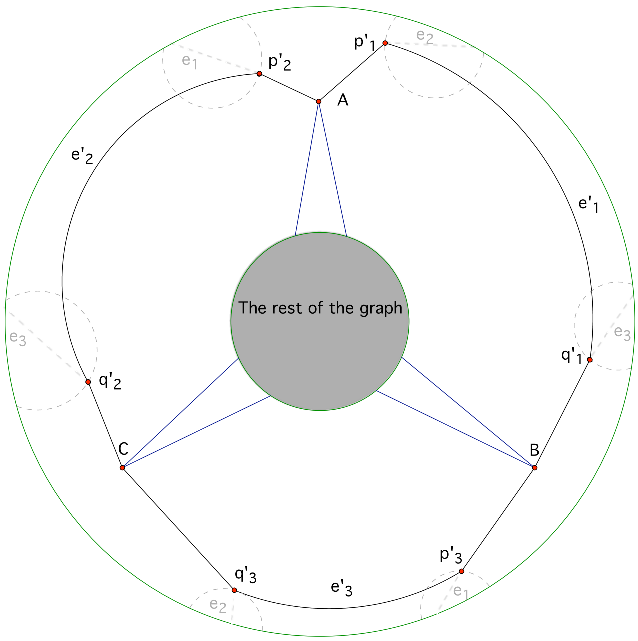


Fig. 5. The same Petersen family graph as in the previous figure with the negative edges re-embedded so that the resulting embedding is affine.

There is one other possible sign class of Petersen family graphs. We say that a signed graph Σ *contains exactly one all-negative* 5-*cycle*, provided all edges are positive except all edges in a 5-cycle are negative. For brevity, we also denote this case as C_5^- in Fig. 7. Before we show that this is the only new sign class of Petersen family graphs, we note that we can perform a $\Delta - Y$ exchange on a signed graph G, where the triangle (A, B, C) is balanced, as illustrated in Fig. 6. We note that by switching if necessary, a signed graph with a Y can always be transformed into a new signed graph by a $Y - \Delta$ exchange.

We use the following:

Lemma 10. *Given a signed graph G_Δ with a balanced triangle (A, B, C) and associated G_Y that results from performing a $\Delta - Y$ exchange on G_Δ on (A, B, C), then the following are true:*

1. *G_Δ is balanced if and only if G_Y is balanced.*
2. *G_Δ contains V_2^o if and only if G_Y contains V_2^o.*

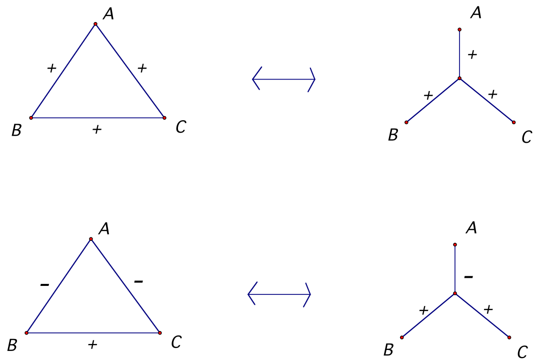


Fig. 6. Allowable triangle-Y exchanges on signed graphs.

3. G_Δ *contains exactly one all-negative* 3*-cycle if and only if* G_Y *contains exactly one all-negative* 3*-cycle.*
4. G_Δ *contains exactly one all-negative* 5*-cycle if and only if* G_Y *contains exactly one all-negative* 5*-cycle.*
5. G_Δ *contains a balancing vertex if and only if* G_Y *contains a balancing vertex.*

Proof. First note that every cycle in G_Δ, except for (A, B, C), corresponds to exactly one cycle in G_Y with the same sign. For cycles disjoint from the cycle (A, B, C), this is clear. A cycle that use exactly one edge of (A, B, C) corresponds to a cycle that passes through two edges of the Y. The same is true for a cycle that use exactly two edges of (A, B, C). If G_Δ is equivalent to a signing with exactly one all-negative 3-cycle, it is impossible for the all-negative 3-cycle to use any edges of (A, B, C): if exactly one edge is used, this would imply that (A, B, C) is not balanced. If exactly two edges are used, this would imply again that (A, B, C) is unbalanced. Similarly, if G_Δ is equivalent to a signing with exactly one all-negative 5-cycle, then the all-negative 5-cycle could not use exactly one edge of (A, B, C). The all-negative 5-cycle could use exactly two edges of (A, B, C), but then there would be a corresponding all-negative 5-cycle in G_Y. □

Note that $K_{4,4} - e$ has no 5-cycles, yet there is a signing for P_7 that has exactly one negative 5-cycle. This is consistent with Lemma 10, as a signing of P_7 with exactly one negative 5-cycle has no balanced triangles.

Finally, we have:

Proposition 11. *All signings of Petersen Family graphs result in one of the following possibilities, up to switching equivalence: balanced,* V_2^o *as a minor, a balancing vertex, contains exactly one all-negative* 3*-cycle, or contains exactly one all-negative* 5*-cycle.*

Proof. We first consider all possible signings of K_6. We use the readily verified fact that every signing is equivalent to a signing for which the number of negative edges at each vertex is less than or equal to half of the degree of the vertex. Then all possible cases are demonstrated in Fig. 7, where the highlighted edges indicate negative edge configurations in K_6. The various configurations are grouped by the resulting sign class of K_6.

Since every Petersen family graph can be obtained from K_6 by $\Delta - Y$ and $Y - \Delta$ exchanges, our proposition follows from Lemma 10. □

We need the following lemma, which follows analogously to the proof for graphs that are linkless in space [7]:

Lemma 12. *Given a linkless signed graph* Σ_Y *and the associated* Σ_Δ *obtained from* G_Y *by a* $\Delta - Y$ *exchange, if* G_Y *is linkless, then so is* G_Δ*.*

Now we show the following:

Proposition 13. *A signed Petersen family graph,* Σ*, containing exactly one all-negative* 5*-cycle has a linkless embedding.*

Proof. First, we illustrate a linkless embedding of the Petersen graph with exactly one all-negative 5-cycle, in Fig. 8.

Now, we claim that every signed Petersen family graph containing exactly one all-negative 5-cycle can be obtained from the classic signed Petersen graph containing exactly one all-negative 5-cycle by $Y - \Delta$ exchanges, with one exception. This can be verified by checking that every 5-cycle in K_6, P_7, and P_8 either contains a chord between vertices of the 5-cycle or a triangle in the edge complement of the 5-cycle, and by checking that the graph P_9 has two classes of 5-cycles. Cycles in one class contain a chord. Cycles in the other class contain neither a chord nor a complementary cycle, but there is a linkless embedding of P_9 with an all-negative cycle in this troublesome class as illustrated in Fig. 9 (and the only triangle in this class of P_9 is unbalanced).

Finally, every signing of K_6, P_7, P_8 and P_9 containing exactly one negative 5-cycle contains a balanced triangle, and thus is obtainable from a Petersen family graph with one more vertex, by a $Y - \Delta$ exchange. We note here that every 5-cycle in K_6 contains a chord, as do both classes of 5-cycles in P_7. The graph P_8 has three classes of 5-cycles; cycles in two classes contain chords, and cycles in the other class contain a complementary triangle.

This concludes our proof. □

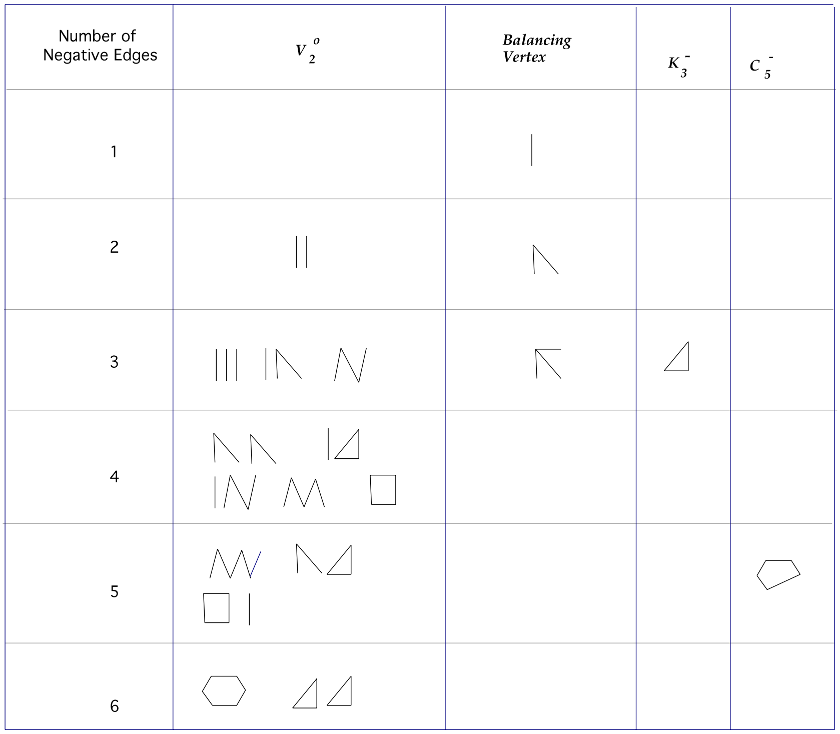


Fig. 7. The possible signings (up to equivalence) of K_6.

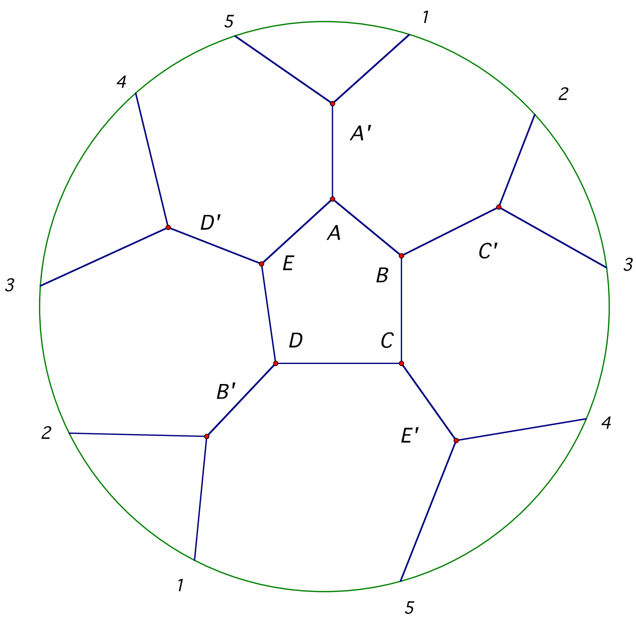


Fig. 8. A linkless embedding of the Petersen graph with exactly one negative 5-cycle.

4. Connectivity and main result

Many of the ideas in this section were inspired by Zaslavsky's work [15].

Proposition 14. *A signed graph has a linkless embedding if and only if each component has a linkless embedding and at most one component is unbalanced.*

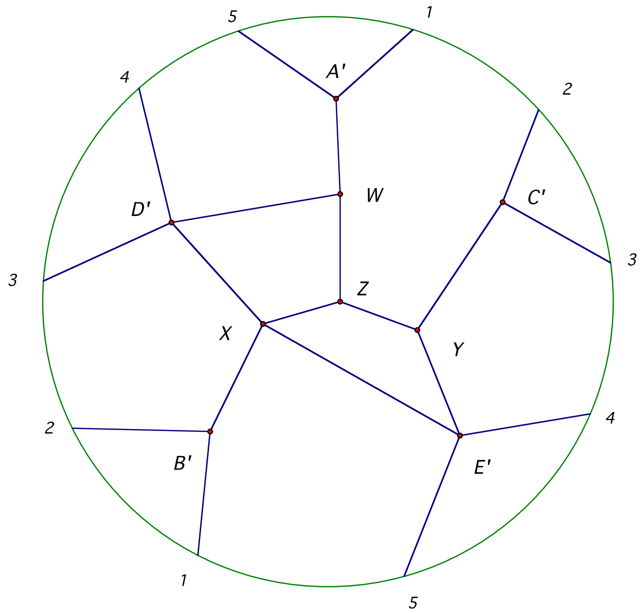


Fig. 9. A linkless embedding of P_9 with exactly one negative 5-cycle, with only triangle unbalanced.

Proof. First, assume a signed graph Σ has a linkless embedding, it follows that each component must also have a linkless embedding. Next, suppose Σ has more than one unbalanced component. Then Σ would contain V_2^o as a minor and be intrinsically linked, a contradiction.

In the other direction, we proceed in cases.

Case 1: *All components are balanced*. By Corollary 5, since each component is balanced and has a projective linkless embedding, then each component also has an associated affine linkless embedding. We can then embed each of these components linklessly, and attain an affine linkless embedding of the entire graph.

Case 2: *Exactly one component is unbalanced*. Embed the unbalanced component linklessly. By Corollary 5, each balanced component has an affine linkless embedding. Embed the balanced components into $\mathbb{R}P^3$ with these affine linkless embeddings so that they do not cross the embedding of the unbalanced component in a projection. Then we have a linkless embedding of the entire graph. □

Corollary 15. *A minor-minimal intrinsically linked signed graph that is not V_2^o is connected.*

We need the following, which is closely related to Proposition 7.

Lemma 16. *A signed graph with a balancing vertex and a linkless embedding has a linkless embedding in which the only edges that cross the boundary are those incident to the balancing vertex.*

Proof. Let (G, σ) be a signed graph with a balancing vertex and a linkless embedding. By Proposition 8, G has a linkless spatial embedding, as it does not contain a Petersen family graph as a minor. The claim then follows from Proposition 7. □

Recall that a *block* of a connected graph G is a maximal 2-connected subgraph of G.

Proposition 17. *A signed graph Σ has a linkless embedding if and only if every block has a linkless embedding and Σ has a balancing vertex if there is more than one unbalanced block.*

Proof. Assume that Σ is a graph that can be embedded linklessly. Clearly an intrinsically linked block cannot exist. Assume that there is more than one unbalanced block. Because Σ cannot contain V_2^o as a subgraph, the negative cycles of the two unbalanced blocks cannot be disjoint, and thus must both share a vertex. Then, as all negative cycles pass through that connecting vertex, it is a balancing vertex.

Now assume that every block in Σ has a linkless embedding and that Σ has a balancing vertex if there is more than one unbalanced block.

Case 1: *There are* 0 *or* 1 *unbalanced blocks*. Embed the unbalanced block linklessly. By Corollary 5 we can find a linkless affine embedding of each of the balanced blocks such that in a projection, edges of one block do not cross edges of any other block. Because the unbalanced block cannot contain V_2^o as a subgraph, there are no disjoint negative cycles in Σ. Also, we may

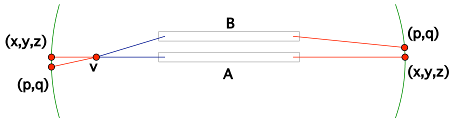


Fig. 10. Blocks with a balancing vertex.

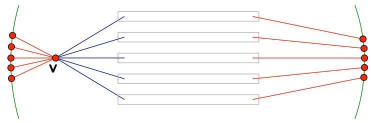


Fig. 11. Blocks with a balancing vertex.

connect the blocks on at most one vertex such that the edges of the balanced block do not cross edges of the other in the projection, and thus construct a linkless embedding of Σ.

Case 2: *There are* 2 *or more unbalanced blocks and a balancing vertex*. Let A be one of the unbalanced blocks. By Lemma 16, we may embed A linklessly with all positive edges disjoint from the boundary and with all negative edges incident to a vertex v and only crossing the boundary once.

Let us take the next unbalanced block, B, and perform the same beginning step. Switch such that all negative edges are incident to a vertex v_1. Embed B such that all of the edges in B cross over the edges in A in a projection (see Fig. 10). Recall that Σ has a balancing vertex, and so the balancing vertex v of A must be the same as the balancing vertex v_1 of B.

As all edges in block B cross over the edges of block A, there can be no linking between the cycles in $A - v$ and cycles in $B - v$. As the two blocks are connected by only one vertex, there is no cycle involving edges in both A and B. Finally, as each block itself is linkless, we know that the embedding of these first two unbalanced blocks creates no new links, and so the embedding of Σ is thus far linkless. One can easily see that embedding any other number of unbalanced blocks in this way maintains this linklessness (see Fig. 11).

As Σ has a balancing vertex, we know that the unbalanced blocks must all share the vertex v in this manner. Now embed the balanced blocks linklessly and affine (which we can do by Corollary 5) such that the balanced blocks do not cross any of the edges from the other blocks in a projection. As these blocks are all linkless as well, no new links are created. Thus, in this case, Σ has a linkless embedding. $\square$

We now may establish the following:

Corollary 18. *No graph with connectivity* 1 *is minor-minimal intrinsically linked.*

Proof. Suppose a minor-minimal intrinsically linked graph, Σ, has connectivity 1. By minimality, every block of Σ has a linkless embedding. By Proposition 17, Σ must have at least two unbalanced blocks. If Σ has no V_2^o subgraph, then a vertex separating two unbalanced blocks must be a balancing vertex. This is a contradiction of Proposition 17 as Σ is intrinsically linked. $\square$

We will show there are no minor-minimal intrinsically linked graphs with connectivity 2 and no new such graphs with connectivity 3.

Let Δ and Γ be graphs such that $\Delta \subseteq \Gamma$. We define a *bridge* of Δ as a maximal subgraph B of Γ such that for any two elements (vertices or edges) of B, there is a path in B containing both and meeting Δ, if at all, only at its endpoints. Following Slilaty [13], we call a signed graph *tangled* if it is unbalanced, has no balancing vertex, and does not include V_2^o as a minor.

Lemma 19. *A tangled minor-minimal intrinsically linked graph is* 3-*connected.*

Proof. Let Σ be a tangled minor-minimal intrinsically linked graph. We know Σ must be 2-connected, by the Corollary 18. Now suppose we can disconnect Σ by removing two vertices $\{v, w\}$. Let b denote an additional unsigned edge (v, w) not in $|\ \Sigma\ |$. For $\epsilon \in \{+, -\}$, let

$$B_\epsilon = \bigcup \{B : B \text{ is a bridge of } \{v, w\} \text{ and } B \cup \epsilon b \text{ is balanced}\}$$

and $\hat{B_\epsilon} = B_\epsilon \cup \epsilon b$; this is balanced. Here we use Lemma 16 from Zaslavsky [15]: because Σ has no balancing vertex, there is exactly one unbalanced bridge of $\{v, w\}$. Let us call this B_0. Now let $\hat{B_0}$ be B_0 with ϵb added if $B_\epsilon \neq \emptyset$ (note: it is possible that both $+b$ and $-b$ will be added).

Now consider the two $\hat{B_\epsilon}$. The added ϵb edges can be replaced by a contracted vw ϵ-path of the appropriate sign from B_0, which is in Σ but outside B_ϵ. We know vw—paths of both signs exist in B_0 since it is unbalanced and 2-connected. Then Σ

also includes a $vw(-\epsilon)$-path in B_0, so $\hat{B_\epsilon}$ is a proper minor of Σ, and hence has a linkless embedding. Consider $\hat{B_0}$. It includes added ϵb edges only if they can be replaced with appropriate vw paths from the B_ϵ subgraphs, so $\hat{B_0}$ is also a proper minor of Σ and can be linklessly embedded.

Embed these three subgraphs linklessly in projective space. Since the $\hat{B_\epsilon}$ are balanced by construction, Corollary 5 implies that we can take their embeddings to be affine. Now replace the ϵb in our embedding of $\hat{B_0}$ with these affine linkless $\hat{B_\epsilon}$ graphs by gluing the $v, w \in V(\hat{B_\epsilon})$ to the v, w endpoints of ϵb, such that each embedded $\hat{B_\epsilon}$ lies within a small tubular neighborhood of ϵb. This gives a linkless embedding of Σ, a contradiction. Hence Σ must be 3-connected. □

Consider three mutually adjacent vertices in a graph G. Remove all edges between these three vertices; $G \therefore$ will be used to refer to this new graph. In general, let $G \therefore$ represent a graph with three marked vertices that are mutually non-adjacent. If $G_1 \therefore$ and $G_2 \therefore$ are two such graphs, then $G_1 \therefore G_2$ is a graph obtained by gluing $G_1 \therefore$ and $G_2 \therefore$ along their three marked vertices, called a 3-*sum*. Note that the resulting graph has connectivity less than or equal to 3. We restrict our 3-sum of signed graphs further by requiring the corresponding triangles to be balanced and the corresponding edges to agree in sign. We define 2-sums and 1-sums of graphs analogously.

We will use the following result from Slilaty [13].

Theorem 20. *If Σ is connected and tangled, then one of the following holds:*

- *Σ is projective planar.*
- *Σ is K_5^-, possibly with positive loops or redundant edges of the same sign.*
- *Σ is a 1-, 2-, or 3-sum of a tangled signed graph and a balanced signed graph with at least 2, 3, or 5 vertices, respectively.*

From Lemma 19, we know that graphs with less than connectivity 3 can be disregarded as possible minor-minimal intrinsically linked signed graphs, so we can disregard 1- and 2-sums of tangled graphs and balanced graphs. We can also disregard sums using K_5^-, as every 3-cycle in K_5^- is unbalanced, and we wish to 3-sum with a balanced graph. Hence any new potential minor minimal intrinsically linked signed graph is a 3-sum of a tangled signed graph with a balanced signed graph. In the following proposition, we prove that no such graph is a new case.

Lemma 21. *Let Σ be a (tangled) signed graph with a linkless embedding and let H^+ be a balanced graph on at least 5 vertices, then $\Sigma \therefore H^+$ is not minor-minimal intrinsically linked, unless $\Sigma \therefore H^+$ is a signed Petersen family graph that is balanced or has a balancing vertex.*

Proof. Let $\Sigma = (G, \sigma)$ be a signed graph with a projective linkless embedding and H^+ a balanced graph with at least 5 vertices. By Lemma 19, we may assume that both G and H are 3-connected. It follows that both G and H have K_4 as a minor. Suppose that $\Sigma \therefore H^+$ is minor-minimal intrinsically linked. Since $\Sigma \therefore K_4+$ is a minor of $\Sigma \therefore H^+$, then $\Sigma \therefore K_4+$ has a linkless embedding. Since Σ can be obtained from $\Sigma \therefore K_4+$ by a $Y - \Delta$ exchange, it follows, by Lemma 12, that Σ has a linkless embedding, and moreover, Σ has a linkless embedding for which the marked triangle, (u, v, w), is affine.

Case 1: *H has a linkless affine embedding.*

Embed Σ linklessly in $\mathbb{R}P^3$ such that the cycle (u, v, w) is not touching the boundary. Deform the embedding so that the cycle lies in the equatorial disk of $\mathbb{R}P^3$.

As H^+ has a linkless affine embedding, by [11] H has a flat spatial embedding (that is, every cycle C bounds a disk D with $H \cap D = \partial D = C$). Moreover, H can be embedded in $\mathbb{R}^3$ such that (u, v, w) lies in the xy-plane and the rest of the graph lies above the plane and projects within the (u, v, w) circle. Similarly, H^+ can be flatly embedded in $\mathbb{R}P^3$, glued to Σ along (u, v, w), such that the rest of H^+ is embedded above the equatorial disk, disjoint from the boundary, and projects within the (u, v, w) circle. By construction, any 2-component link with one component contained in H and the other contained in G would be splittable.

Consider this embedding of $\Sigma \therefore H^+$, with the three edges of the attaching triangle included. We claim there are no non-split links in the resulting embedding. By construction, any possible non-split 2-component link, $C_1 \cup C_2$, would have to share vertices and edges of G and H. One cycle, say C_1, must be entirely contained in H or entirely contained in G.

Case 1a: *C_1 is entirely contained in H.*

In this case, C_2 can be written as the 2-sum of a cycle contained in H and a cycle contained in G, neither of which forms a non-splittable link with C_1. Denote these two cycles as K and L, with $K \subset H$ and $L \subset G$. Since H^+ is flatly embedded, it follows that there is a 3-ball, B such that $K \subset B$ and $B \cap C_1 = \emptyset$. Further, there exists such a B with $B \cap (K \cup L) = K$. It follows that (C_1, C_2) is ambient isotopic to (C_1, L), and thus (C_1, C_2) is a splittable link.

Case 1b: *C_2 is entirely contained in G.*

In this case, C_2 can be written as the 2-sum of a cycle contained in H and a cycle contained in G, neither of which forms a non-splittable link with C_1. Denote these two cycles as K and L, with $K \subset H$ and $L \subset G$. In the given embedding, K is contained in a 3-ball, B, such that $B \cap \Sigma = e$, where e is the edge shared by K and L. It follows that (C_1, C_2) is ambient isotopic to (C_1, L), which is splittable.

Thus $\Sigma \therefore H^+$ is not intrinsically linked in this case.

Case 2: *H is intrinsically linked*. In this case, H contains a Petersen family graph as a minor [11]. Then $\Sigma \therefore H^+$ contains $(K_4, \sigma) \therefore H^+$ as a minor, where σ is some signing for K_4. Since H contains a Petersen family graph as a minor, and the underlying graph of $(K_4, \sigma) \therefore H^+$ is also obtainable from H by a $\Delta - Y$ exchange, it follows that $(K_4, \sigma) \therefore H^+$ contains a signed Petersen family graph as a minor. Moreover, since (K_4, σ) must contain a balanced triangle, it follows that $(K_4, \sigma) \therefore H^+$ contains a signed Petersen family graph as a minor, where the signed Petersen family graph is either balanced or contains a balancing vertex. By minor-minimality, it follows that $\Sigma \therefore H^+$ must be a signed Petersen family graph that is either balanced or contains a balancing vertex. □

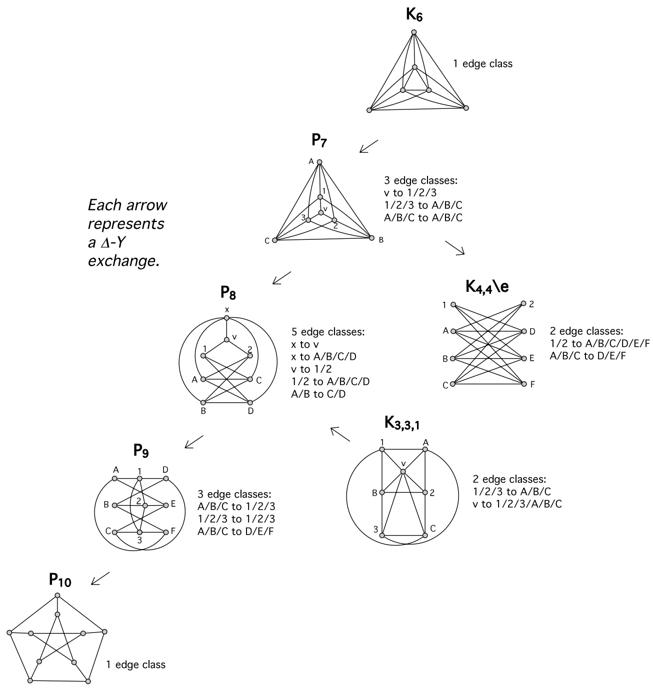
Finally, we may state our main result.

Theorem 22. *The graphs V_2^o, seven balanced Petersen family graphs,* 32 *Petersen family graphs with a balancing vertex, and six Petersen family graphs containing exactly one all-negative* 3*-cycle form the complete set of minor-minimal intrinsically linked signed graphs in projective space.*

Acknowledgments

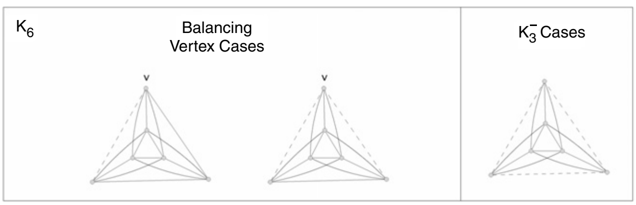
We would like to thank Kenji Kozai and the referee for making valuable suggestions.

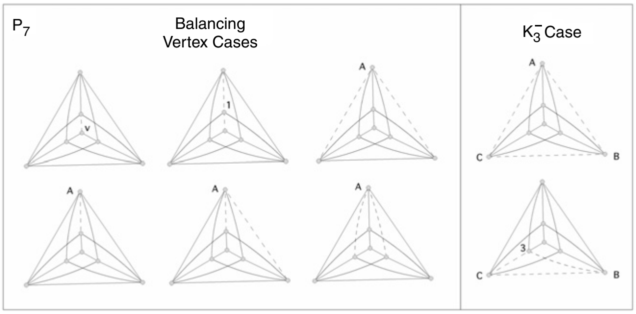
Appendix A. Petersen family graphs and their edge classes

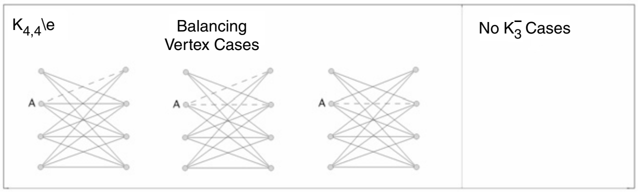


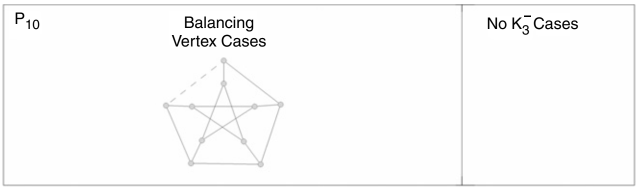
Appendix B. Balancing vertices and K_3^- cases

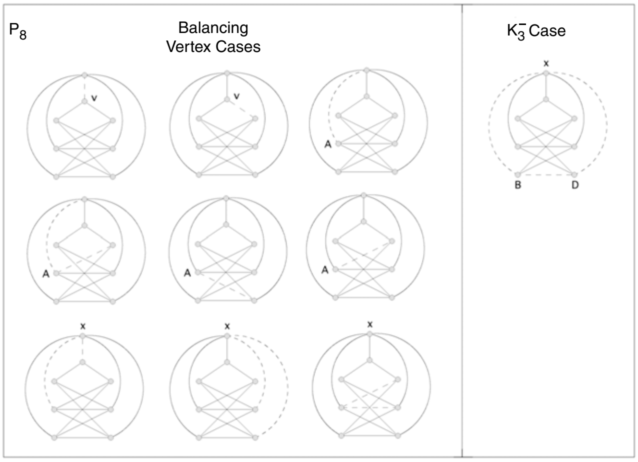
We adopt the convention of having a dashed line represent a negatively signed edge while the solid edges are positively signed. Here, we demonstrate the distinct balancing vertex cases of each Petersen family graph as well as the distinct Petersen family graphs with all positive edges except one all-negative 3-cycle.

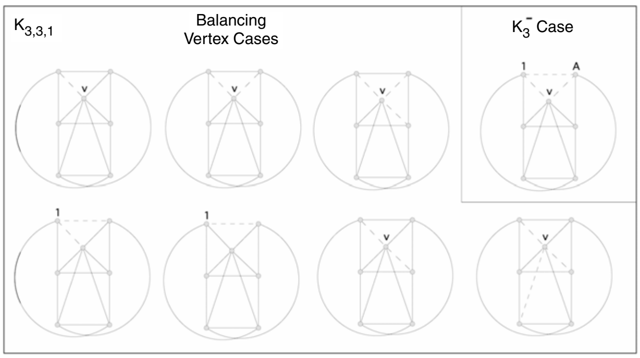


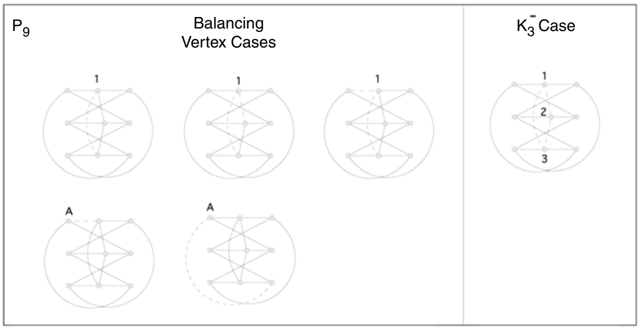

P_7
Balancing
Vertex Cases
v
1
A
A
A
A
K_3^- Case
A
C
B
3
C
B


$K_{4,4}\backslash e$
Balancing
Vertex Cases
A
A
A
No K_3^- Cases


P_{10}
Balancing
Vertex Cases
No K_3^- Cases


P_8
Balancing
Vertex Cases
v
v
A
A
A
A
x
x
x
K_3^- Case
x
B
D


$K_{3,3,1}$
Balancing
Vertex Cases
v
v
v
1
1
v
v
K_3^- Case
1
A
v



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